① Quadratic inequality $\iff$ LMI

Statement: $x^T P x + q^T x + r \le x^T P' x + q'^T x + r' \quad \forall x.$

$$\iff \underbrace{\begin{bmatrix} P & \frac{1}{2}q \\ \frac{1}{2}q^T & r \end{bmatrix}}_{\bar{P}} \preceq \underbrace{\begin{bmatrix} P' & \frac{1}{2}q' \\ \frac{1}{2}(q')^T & r' \end{bmatrix}}_{\bar{P}'}$$

Pf. $\Leftarrow$ Obvious.

By definition $z^T \bar{P} z \le z \bar{P}' z \quad \forall z$, so pick $z = (x, 1)$.

$\Rightarrow$ It suffices to show that

$$x^T P x + y q^T x + y^2 r \le x^T P' x + y (q')^T x + y^2 r'$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}$.

Pf. If $y \neq 0$, then

$$\left(\frac{x^T}{y}\right)^T P \left(\frac{x}{y}\right) + q^T\left(\frac{x}{y}\right) + r \le \left(\frac{x}{y}\right)^T P' \left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)(q')^T x + r'$$

$$\iff \quad x^T P x + (q^T x) y + y^2 r \le \ldots$$

if $y = 0$ then:

$$x^T P x \le x^T P' x.$$

(By contrapositive: if $x^T P x > x^T P' x$ for some $x \in \mathbb{R}^n$, then let $\omega = tx$ to get

$$t^2 \left(x^T P x - x^T P' x\right) + t(q - q')^T x + (r - r') \to \infty$$

$> 0.$

as $t \uparrow \infty$).

$\blacksquare$